

# THE GERBY GOPAKUMAR-MARIÑO-VAFA FORMULA

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ABSTRACT. We prove a formula for certain cubic  $\mathbb{Z}_n$ -Hodge integrals in terms of loop Schur functions. We use this identity to prove the Gromov-Witten/Donaldson-Thomas correspondence for local  $\mathbb{Z}_n$ -gerbes over  $\mathbb{P}^1$ .

## 1. INTRODUCTION

**1.1. Statement of Results.** The Gopakumar-Mariño-Vafa formula, proven independently in [LLZ04] and [OP04], evaluates certain generating functions of cubic Hodge integrals on moduli spaces of curves in terms of Schur functions, a special basis of the ring of symmetric functions. The formula can be interpreted as one instance of the Gromov-Witten/Donaldson-Thomas correspondence for Calabi-Yau (CY) 3-folds. In this paper, we generalize the Gopakumar-Mariño-Vafa formula to  $\mathbb{Z}_n$ -Hodge integrals and we show that this formula can be viewed as one instance of the orbifold GW/DT correspondence.

In particular, we define generating functions  $\tilde{V}_\mu^\bullet(a)$  of cubic  $\mathbb{Z}_n$ -Hodge integrals on moduli spaces of stable maps to the classifying space  $\mathcal{B}\mathbb{Z}_n$ . These generating functions are indexed by conjugacy classes  $\mu$  of the generalized symmetric group  $\mathbb{Z}_n \wr S_d$  and are closely related to the GW orbifold vertex developed in [Ros11]. In place of the Schur functions in the usual Gopakumar-Mariño-Vafa formula, we introduce generating functions  $\tilde{P}_\lambda(a)$  which are specializations of *loop* Schur functions, developed in [LP08] and [Ros12]. These generating functions are indexed by irreducible representations  $\lambda$  of  $\mathbb{Z}_n \wr S_d$  and are closely related to the DT orbifold vertex developed in [BCY10]. The main result is the following correspondence via the character values  $\chi_\lambda(\mu)$  of  $\mathbb{Z}_n \wr S_d$ .

**Theorem 1.** *After an explicit change of variables,*

$$\tilde{V}_\mu^\bullet(a) = \sum_{\lambda} \tilde{P}_\lambda(a) \frac{\chi_\lambda(\mu)}{z_\mu}$$

There are  $n$  distinct  $\mathbb{Z}_n$ -gerbes  $\mathcal{G}_k$  ( $0 \leq k < n$ ) over  $\mathbb{P}^1$  classified by  $H^2(\mathbb{P}^1, \mathbb{Z}_n)$ . We define  $\mathcal{X}$  to be a *local  $\mathbb{Z}_n$ -gerbe over  $\mathbb{P}^1$*  if  $\mathcal{X}$  is isomorphic to the total space of a rank two Calabi-Yau orbifold bundle over some  $\mathcal{G}_k$ . Applying the gluing rules of [Ros11] and [BCY10], Theorem 1 leads to a proof of the orbifold GW/DT correspondence for local  $\mathbb{Z}_n$ -gerbes over  $\mathbb{P}^1$ .

**Theorem 2.** *After an explicit change of variables, the GW potential of any local  $\mathbb{Z}_n$ -gerbe over  $\mathbb{P}^1$  is equal to the reduced, multi-regular DT potential.*

This is the first example of the GW/DT correspondence for orbifold targets with nontrivial curve classes contained in the singular locus.

**1.2. Context and Motivation.** Atiyah-Bott localization ([AB84]) has proven to be an extremely powerful tool in both GW and DT theory of toric CY 3-folds. In particular, it has led to the development of the (orbifold) *topological vertex* ([AKMV05, ORV06, LLLZ09, BCY10, Ros11]), a basic building block for the GW or DT theory of all toric CY 3-folds. In the GW case the vertex is a generating function of (abelian) Hodge integrals, whereas in the DT case the vertex can be defined purely in terms of combinatorics.

The topological vertex formalism provides us with an algorithm for proving conjectural correspondences related to GW and DT theory: first prove that the correspondence holds locally for the vertex, then show that it is consistent with the gluing laws. In the smooth case, this approach was utilized to prove the GW/DT correspondence for toric 3-folds, beginning with [LLZ04, MLZ07, LLLZ09] and concluding with [MOOP08].

In orbifold Gromov-Witten theory, the first example of this local-to-global approach appeared in [CR11] where it was used to prove an example of the crepant resolution conjecture. It was further developed in [Ros11], where a correspondence between the  $A_n$  GW and DT vertex theories was suggested. In [Zon11], this correspondence was proven for the effective one-leg  $A_n$  vertex. One consequence of Zong’s result is the orbifold GW/DT correspondence for local footballs (orbifolds with coarse space  $\mathbb{P}^1$  and smooth away from 0 and  $\infty$ ). The main focus of this paper is the *ineffective* one-leg  $A_n$  vertex.

In the ineffective case, several new challenges arise. On the GW side one can no longer utilize the  $\mathbb{Z}_n$ -Mumford relation which was the key tool in [Zon11]. Moreover, the orbifold structure at the nodes of the source curve is no longer determined by the degree of the corresponding map. As a consequence the invariants are no longer indexed by partitions, but by decorated partitions. The first challenge was overcome in [Zon12], where a set of linear relations for the vertex was proven to be deterministic. As suggested in [Ros11], the latter challenge is overcome by encoding the twisted partitions as conjugacy classes in the generalized symmetric group  $\mathbb{Z}_n \wr S_d$ . The key to the correspondence between the GW and DT vertex then lies in the associated character table.

On the DT side, the effective case can be interpreted in terms of Schur functions, but we lose this interpretation when we pass to the ineffective case. However, we observe that the DT vertex can naturally be interpreted as specializations of *loop* Schur functions, developed in [LP08] and [Ros12]. The combinatorial structure of the loop Schur functions provides us in turn with very useful properties of the DT vertex which are integral in the arguments of this paper.

Many interesting questions arise from this work. First, the results of this paper give the first example of the orbifold GW/DT correspondence for a target which contains nontrivial curve classes which lie entirely in the singular locus. In order to state the GW/DT correspondence in this case, it is necessary to discard a significant amount of information on the DT side by restricting to the multi-regular contributions. It would be interesting to generalize orbifold GW theory to account for this extra data and one possible approach seems to lie in the *very twisted stable maps* of [CMU10]. Secondly, since the current work completes the one-leg  $A_n$  GW/DT correspondence, another natural extension of this work is to extend the results herein to the two, and ultimately the three-leg  $A_n$  vertex. Finally, the  $A_n$  vertex is by far the easiest geometry in both GW and DT theory. It would be extremely interesting to study if/how the GW/DT vertex correspondence extends to noncyclic and/or nonhard-lefschetz orbifolds.

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**1.4. Plan of the Paper.** After setting up notation and giving a precise statement of Theorems 1 and 2 in Section 2, we study the geometry of the framed GW vertex  $\tilde{V}_\mu^\bullet(a)$  in Section 3. In particular, we develop a set of bilinear relations which were shown in [Zon12] to uniquely determine the GW vertex from a certain generating function of rubber integrals. In Section 3.4, we interpret these rubber integrals in terms of wreath Hurwitz numbers and apply the Burnside formula to write the bilinear relations in terms of the characters of the generalized symmetric group  $\mathbb{Z}_n \wr S_d$ . In Section 4, we recall the definition of loop Schur functions and the main results from [Ros12] which are pivotal in the proof of Theorem 1. We also recall a hook-length formula from [ER88] and [Nak09] which relates the loop Schur functions to the framed DT vertex  $\tilde{P}_\lambda(a)$ . In Section 5, we study the representation theory of  $\mathbb{Z}_n \wr S_d$  where the main tool is the wreath Fock space. Finally, in Section 6 we put everything together to prove Theorem 1. In Section 7 we use gluing rules developed [Ros11] and [BCY10] to show how the GW/DT correspondence for local  $\mathbb{Z}_n$ -gerbes over  $\mathbb{P}^1$  follows from Theorem 1.

## 2. BACKGROUND AND NOTATION

In this section we set up notation which will be used throughout the paper and we give a precise statement of the main results.

**2.1. Partitions.** For each positive integer  $n$  we fix a generator of the cyclic group

$$\mathbb{Z}_n = \langle \xi_n := e^{\frac{2\pi\sqrt{-1}}{n}} \rangle.$$

When no confusion arises, we write the generator simply as  $\xi$ . It is well known that  $n$ -tuples of partitions naturally correspond to conjugacy classes and irreducible representations of  $\mathbb{Z}_n \wr S_d$ , see e.g. [Mac95]. We will use  $\mu$ ,  $\nu$ , and  $\tau$  to denote  $n$ -tuples of partitions corresponding to conjugacy classes and reserve  $\lambda$  and  $\sigma$  to refer to irreducible representations. We let  $\chi_\lambda(\mu)$  denote the value of the character of the irreducible representation  $\lambda$  on the conjugacy class  $\mu$ .

Consider the  $n$ -tuple of partitions

$$\mu = \left( (d_1^0, \dots, d_{l_0}^0), \dots, (d_1^{n-1}, \dots, d_{l_{n-1}}^{n-1}) \right)$$

with  $d_j^i \in \mathbb{N}$ . Let  $\mu^i = (d_1^i, \dots, d_{l_i}^i)$  denote the partition indexed by  $i$  and let  $\mu'$  correspond to the  $n$ -tuple of *twisted* partitions  $(\emptyset, \mu^1, \dots, \mu^{n-1})$ . At times it will be convenient to write  $\mu$  as a vector  $(\dots \xi^i d_j^i \dots)$  where the power of  $\xi$  keeps track of which  $\mu^i$  the  $d_j^i$  came from. Let  $l(\mu) := \sum l_i$  denote the length of  $\mu$ . Set  $|\mu^i| := \sum_j d_j^i$  and  $|\mu| := \sum |\mu^i|$ . Let  $\underline{\mu}$  denote the underlying partition of  $\mu$  that forgets the  $\mathbb{Z}_n$  decorations. We define  $-\mu := (\dots, \xi^{n-i} d_j^i, \dots)$  to be the  $n$ -tuple of partitions with opposite twistings. We also define

$$z_\mu := |\text{Aut}(\mu)| \prod n d_j^i$$

to be the order of the centralizer of any element in the conjugacy class of  $\mu$ .

Suppose  $\lambda = (\lambda_0, \dots, \lambda_{n-1})$ . Via  $n$ -quotients (described explicitly in Section 5.2)  $\lambda$  can be identified with a partition of  $nd$  where  $d = \sum |\lambda_i|$ . We denote this corresponding partition by  $\bar{\lambda}$ . We write  $\bar{\lambda} = \{(i, j)\}$  where  $i$  indexes the rows and  $j$  indexes the columns of the Young diagram corresponding to  $\bar{\lambda}$ . We will often think of  $\bar{\lambda}$  as a colored Young diagram where the box  $(i, j)$  has color  $j - i \pmod n$ . We denote the boxes with color  $k$  by  $\bar{\lambda}[k]$ . For  $\square \in \bar{\lambda}$ , we let  $h_k(\square)$  denote the number of color  $k$  boxes in the hook defined by  $\square$  and we define

$$n_k(\bar{\lambda}) := \sum_i (i-1) (\# \text{ of color } k \text{ boxes in the } i\text{th row}).$$

We let  $\gamma$  denote a tuple of *nontrivial* elements in  $\mathbb{Z}_n$ . We define  $m_i(\gamma)$  to be the number of occurrences of  $\xi^i \in \mathbb{Z}_n$  in  $\gamma$ .

**2.2. Gromov-Witten Theory.** Given  $\mu$  and  $\gamma$  as above, let  $\overline{\mathcal{M}}_{g, \gamma + \mu}(\mathcal{B}\mathbb{Z}_n)$  denote the moduli stack of stable maps to the classifying space with  $m_i(\gamma) + l_i(\mu)$  marked points twisted by  $\xi^i$ .

For any  $a \in \frac{1}{n}\mathbb{Z}$ , the special cubic Hodge integrals we are interested in are

$$(1) \quad V_{g,\gamma}(\mu; a) := \frac{a^{l_0}}{|\text{Aut}(\mu)|} \prod_{i=0}^{n-1} \prod_{j=1}^{l_i} \frac{\prod_{k=1}^{d_j^i} (ad_j^i - \frac{i}{n} + k)}{(-1)^{d_j^i - \delta_{i,0}} d_j^i \cdot d_j^i!} \int_{\mathcal{M}_{g,\gamma+\mu}(\mathcal{B}\mathbb{Z}_n)} \frac{\Lambda^0(1) \Lambda^\xi(a) \Lambda^{\xi-1}(-a-1)}{\delta(a) \prod_{i=0}^{n-1} \prod_{j=1}^{l_i} \left( \frac{1}{d_j^i} - \psi_{i,j} \right)}$$

where

$$\Lambda^\xi(t) := (-1)^{rk(\mathbb{E}^\xi)} \sum_{i=1}^{rk(\mathbb{E}^\xi)} (-t)^{rk(\mathbb{E}^\xi)-i} c_i(\mathbb{E}^\xi)$$

and  $\delta(a)$  is the function which takes value  $a^2+a$  on the connected component of the moduli space which parametrizes trivial covers of the source and takes value 1 on all other components.

Introduce formal variables,  $u$  and  $x_i$  to track genus and marks. Also introduce the variables  $p_\mu$  with formal multiplication defined by concatenating the indexing partitions. Then we define

$$V_\mu^\bullet(x, u; a) := \exp \left( \sum_{g,\gamma,\nu} V_{g,\gamma}(\nu; a) u^{2g-2+l(\nu)} \prod_{i=1}^{n-1} \frac{x_i^{m_i(\gamma)}}{m_i!} p_\nu \right) [p_\mu]$$

where  $[p_\mu]$  denotes “the coefficient of  $p_\mu$ ”. By definition,  $V_\mu^\bullet(x, u; a)$  is the one-leg  $A_{n-1}$  orbifold GW vertex defined in [Ros11].

For later notational convenience, we define

$$(2) \quad \tilde{V}_\mu^\bullet(a) := \prod_{i=1}^n (\sqrt{-1} \xi_{2n}^i)^{l_i} V_\mu^\bullet(x, u; a).$$

where  $l_n := l_0$ .

**2.3. Donaldson-Thomas Theory.** Let  $q_0, \dots, q_{n-1}$  be formal variables (always assume that the index of  $q_k$  is computed modulo  $n$ ) and define  $q := q_0 \dots q_{n-1}$ . For  $\bar{\lambda}$  as above, define

$$P_\lambda(q_0, \dots, q_{n-1}) := \frac{1}{\prod_{\square \in \bar{\lambda}} \left( 1 - \prod_i q_i^{h_i(\square)} \right)}.$$

By definition,  $P_\lambda(-q_0, \dots, q_{n-1})$  is the reduced one-leg  $A_{n-1}$  orbifold DT vertex defined in [BCY10]. We define the framed DT vertex by

$$(3) \quad \tilde{P}_\lambda(a) := \left( \left( (-\xi_{2n})^{|\lambda|} \prod \xi_n^{l|\lambda_i|} \right)^n \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} \right)^{-a} \frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\lambda)} q^{\frac{d}{2}} (-1)^d \prod_i q_i^{n_i(\bar{\lambda})} P_\lambda(q_0, \dots, q_{n-1}).$$

**Remark 2.1.**  $\chi_{\bar{\lambda}}$  is a character of  $S_{dn}$  whereas  $\dim(\lambda)$  is the dimension of an irreducible representation of  $\mathbb{Z}_n \wr S_d$ . As we will see in Section 5.4, the quotient  $\frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\lambda)}$  is simply a compact way of keeping track of a sign.

**Remark 2.2.** In Corollary 4.4, we relate  $\tilde{P}_{\lambda}(0)$  to loop Schur functions.

**2.4. The Correspondence.** We will prove the following formula.

**Theorem 1.** *After the change of variables*

$$q \rightarrow e^{\sqrt{-1}u}, \quad q_k \rightarrow \xi_n^{-1} e^{-\sum_i \frac{\xi_n^{-ik}}{n} (\xi_{2n}^i - \xi_{2n}^{-i}) x_i},$$

$$\tilde{V}_{\mu}^{\bullet}(a) = \sum_{\lambda} \tilde{P}_{\lambda}(a) \frac{\chi_{\lambda}(\mu)}{z_{\mu}}$$

In Section 7, we use Theorem 1 to deduce the Gromov-Witten/Donaldson-Thomas correspondence for local  $\mathbb{Z}_n$ -gerbes over  $\mathbb{P}^1$ .

**Theorem 2.** *Let  $\mathcal{X}$  be a local  $\mathbb{Z}_n$ -gerbe over  $\mathbb{P}^1$  and let  $GW(\mathcal{X})$  and  $DT'_{mr}(\mathcal{X})$  denote the GW potential and the reduced, multi-regular DT potential of  $\mathcal{X}$ , respectively. After the change of variables*

$$q \rightarrow -e^{\sqrt{-1}u}, \quad q_k \rightarrow \xi_n^{-1} e^{-\sum_i \frac{\xi_n^{-ik}}{n} (\xi_{2n}^i - \xi_{2n}^{-i}) x_i},$$

$$GW(\mathcal{X}) = DT'_{mr}(\mathcal{X}).$$

### 3. GEOMETRY

In this section we set up auxiliary integrals on moduli spaces of relative maps into  $\mathbb{P}^1$ -gerbes in order to obtain bilinear relations between the vertex  $\tilde{V}_{\mu}^{\bullet}(a)$  and certain rubber integrals  $\tilde{H}_{\nu,\mu}^{\bullet}(a)$ . These relations were developed independently in [Zon12] where it was shown that they uniquely determine  $\tilde{V}_{\mu}^{\bullet}(a)$  from  $\tilde{H}_{\nu,\mu}^{\bullet}(a)$ . Since our conventions and notations differ slightly, we reproduce the computations here. The rubber integrals in  $\tilde{H}_{\nu,\mu}^{\bullet}(a)$  can be interpreted as wreath Hurwitz numbers, and we conclude the section by writing  $\tilde{H}_{\nu,\mu}^{\bullet}(a)$  in terms of characters of the generalized symmetric group.

**3.1. Auxiliary Integrals.** Here we set up integrals on moduli spaces of relative stable maps to  $\mathbb{P}^1$ -gerbes. For each line bundle  $\mathcal{O}(-k)$  with  $0 \leq k < n$ , we can define a  $\mathbb{P}^1$ -gerbe  $\mathcal{G}_k$  with isotropy group  $\mathbb{Z}_n$  and an orbifold line bundle  $L_k$  as follows. The gerbe  $\mathcal{G}_k$  is defined by pullback

$$\begin{array}{ccc} \mathcal{G}_k & \longrightarrow & \mathcal{BC}^* \\ \downarrow & & \downarrow \lambda \rightarrow \lambda^n \\ \mathbb{P}^1 & \xrightarrow{\mathcal{O}(-k)} & \mathcal{BC}^* \end{array}$$

and the top map parametrizes the line bundle  $L_k$ . Note that the numerical degree of  $L_k$  is  $-k/n$  and the action of  $\mathbb{Z}_n$  on the fibers is given by multiplication by  $\xi_n$  (see e.g. Section 2.3 of [Ros11]).

We consider the relative moduli spaces  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_k, \mu[\infty])$  which parametrize maps with fixed ramification (and isotropy) profile over  $\infty$ . These moduli spaces were developed in [AF11]. The integrals we will investigate are the following.

$$(I-1) \quad \frac{1}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0, \mu[\infty])} e(R^1\pi_*((\hat{f}^*L_0)(-D) \oplus \hat{f}^*L_0^\vee(-1)))$$

where  $D$  is the locus of relative points on the universal curve with trivial isotropy and  $\hat{f}$  contracts the degenerated target and maps all the way to  $\mathcal{G}_0$ , and

$$(I-2) \quad \frac{1}{|\text{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_k, \mu[\infty])} e(R^1\pi_*(\hat{f}^*L_k \oplus \hat{f}^*L_k^\vee(-1))).$$

**3.2. Partial Evaluations.** In certain cases, we can evaluate the integrals (I-1) and (I-2) explicitly. We collect these computations here.

We begin with the first integral. As we will see in Section 3.3, (I-1) is equal to  $V_{g,\gamma}(\mu; 0)$ . Therefore, we consider special choices of  $\mu$  for which we can evaluate  $V_{g,\gamma}(\mu; 0)$ .

**Lemma 3.1.**

$$V_{g,\gamma}((d); 0) = \delta_{|\gamma|,0} \frac{(-1)^{d-1}}{n} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g(d\psi)^{2g-2}.$$

*Proof.* By (1),  $V_{g,\gamma}((d); 0)$  vanishes away from the locus of maps which parametrize trivial covers. In particular, since  $\gamma$  consists of nontrivial elements in  $\mathbb{Z}_n$ , the cover can only be trivial if  $\gamma = \emptyset$ . On the locus of maps which parametrize trivial covers,  $\mathbb{E}^\xi \cong \mathbb{E}^0$ . Therefore we can apply the Mumford relation to the integrand in the definition of  $V_{g,\emptyset}((d); 0)$ . The lemma follows by pushing forward to  $\overline{\mathcal{M}}_{g,1}$  which is a degree  $\frac{1}{n}$  map.  $\square$

**Corollary 3.2.**

$$V_{(d)}(0) = \frac{(-1)^{d-1}}{2nd} \csc\left(\frac{du}{2}\right).$$

*Proof.* This follows from Lemma 3.1 and [FP00].  $\square$

**Corollary 3.3.**

$$V_\mu^\bullet(0) = \left( \frac{1}{z_{\mu^0}} \prod_{j=1}^{l_0} \frac{(-1)^{d_j^0-1}}{2} \csc\left(\frac{d_j^0 u}{2}\right) \right) V_{\mu'}^\bullet(0)$$

*Proof.* By (1), the only nonzero vertex terms  $V_{g,\gamma}(\mu)$  with  $\mu^0 \neq \emptyset$  are those with a single untwisted node, these invariants were computed in Lemma 3.1. Passing from the connected invariants to the disconnected ones by exponentiating proves the result.  $\square$

From these evaluations, we see that the  $a = 0$  vertex is completely determined once we know the contributions coming from partitions  $\mu$  with  $\mu^0 = \emptyset$ .

For the integral (I-2), we obtain the following vanishing result.

**Lemma 3.4.** *The integral (I-2) vanishes if any of the parts of  $\mu$  are un-twisted.*

*Proof.* The integral vanishes by dimensional reasons. The dimension of the moduli space is  $|\mu| + 2g - 2 + |\gamma| + l(\mu)$ . The degree of the integrand can be computed by orbifold Riemann-Roch, we obtain  $|\mu| + 2g - 2 + |\gamma| + l(\mu')$ .  $\square$

**3.3. Bilinear Relations.** We now compute the integrals (I-1) and (I-2) via localization. We begin with (I-1) by equipping the target with a torus action and choosing an equivariant lift of the integrand with the following weights:

	$T(-\infty)$	$L_0$	$L_0^\vee(-1)$
0	1	$a$	$-a - 1$
$\infty$	0	$a$	$-a$

Each fixed locus of the torus action can be encoded by a bipartite graph  $\Gamma$  with white (black) vertices corresponding to the connected components of  $\hat{f}^{-1}(0)$  ( $\hat{f}^{-1}(\infty)$ ). The vertices and edges are decorated with the following data:

- Each vertex  $v$  is labeled with a tuple  $\gamma_v$  of nontrivial elements in  $\mathbb{Z}_n$  corresponding to the twisted marks on that component and an integer  $g_v$  corresponding to the genus.
- Each edge  $e$  is labeled with a complex number  $(\xi^{k_e} d_e)$  which induces a  $n$ -tuple of partitions  $\nu_v \in \text{Conj}(\mathbb{Z}_n \wr S_{d_v})$  at each white vertex and  $-\nu_v \in \text{Conj}(\mathbb{Z}_n \wr S_{d_v})$  at each black vertex.
- In addition, each black vertex is labeled with a  $n$ -tuple of partitions  $\mu_v$  such that  $|\mu_v| = |\nu_v|$  and the union of all  $\mu_v$  is  $\mu$ .

To a white vertex, we associate the contribution

$$\text{Cont}(v) = V_{g_v, \gamma_v}(\nu_v; a)$$

and to a black vertex we associate the contribution

$$\text{Cont}(v) = \frac{(-1)^{l(\nu^0) + g - 1 + \sum_{i \neq 0} \frac{n-i}{n} (m_i(\gamma_v) + l_i(\mu_v) + l_{n-i}(\nu_v))} (a)^{2g_v - 2 + |\gamma| + l(\mu_v) + l(\nu_v)}}{|\text{Aut}(\nu_v)|} \cdot \left( \prod_{i=1}^{l(\nu_v)} n d_i \right) \int_{\mathcal{M}_{g_v, \gamma_v}(\mathcal{G}_0; \nu_v[0], \mu_v[\infty]) / \mathbb{C}^*} -(-\psi_0)^{2g_v - 3 + |\gamma_v| + l(\nu_v) + l(\mu_v)},$$

where  $\psi_0$  is the target psi class. By the localization formula (see for example [Liu11] or Section 4 of [Zon12]) we compute the integral

$$(I-1) = \frac{1}{|\text{Aut}(\mu)|} \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \prod_v \text{Cont}(v).$$



**Remark 3.5.** In the simplification of the black vertex contribution, we used the  $\mathbb{Z}_n$ -Mumford relation ([BGP08]), namely:

$$\Lambda^\xi(a)\Lambda^{\xi^{-1}}(-a) = a^{rk(\mathbb{E}^\xi)}(-a)^{rk(\mathbb{E}^{\xi^{-1}})}$$

where the ranks can be computed by orbifold Riemann-Roch.

Setting  $a = 0$ , we observe that the contributions from black vertices vanish and the integral is equal to  $V_{g,\gamma}(\mu; 0)$ .

Define the rubber integral generating function

$$H_{\nu,\mu}(x, u) := \frac{1}{|\text{Aut}(\nu)||\text{Aut}(\mu)|} \sum_{g,\gamma} \int_{\overline{\mathcal{M}}} \psi_0^{r+|\gamma|-1} u^r \frac{x_1^{m_1(\gamma)}}{m_1(\gamma)!} \cdots \frac{x_{n-1}^{m_{n-1}(\gamma)}}{m_{n-1}(\gamma)!}$$

where  $r := 2g - 2 + l(\mu) + l(\nu)$  and  $\overline{\mathcal{M}}$  is the space of relative maps into the rubber target:  $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0; \nu[0], \mu[\infty])/\mathbb{C}^*$ .

For notational convenience, we define

$$\tilde{H}_{\nu,\mu}^\bullet(a) := \exp(H_{\nu,\mu}(a\xi_{2n}^{-1}x_1, \dots, a\xi_{2n}^{1-n}x_{n-1}, \sqrt{-1}au))$$

The above localization computations amount to the following bilinear relations between  $V$  and  $H$ :

$$(R-1) \quad \tilde{V}_\mu^\bullet(0) = \sum_{|\nu|=|\mu|} \tilde{V}_\nu^\bullet(a) z_\nu \tilde{H}_{-\nu,\mu}^\bullet(a).$$

We also compute (I-2) via localization using the torus action linearized with weights

	$T(-\infty)$	$L_k$	$L_k^\vee(-1)$
0	1	0	-1
$\infty$	0	$k/n$	$-k/n$

The localization computation is almost identical, however the components mapping onto  $\mathcal{G}_k$  are slightly different. In particular, if a rational curve maps to  $\mathcal{G}_k$  with degree  $d$  and twisting  $i \in \mathbb{Z}_n$  at 0, then the twisting at  $\infty$  must be  $-dk - i \in \mathbb{Z}_n$ . Define  $g_k(\xi^i d) := (\xi^{dk-i} d)$  and naturally extend the definition of  $g_k$  to  $\mu$ . Then localizing (I-2) leads to the relations

$$(R-2) \quad 0 = \sum_{|\nu|=|\mu|} \tilde{V}_\nu^\bullet(0) z_\nu \tilde{H}_{g_k(\nu),\mu}^\bullet\left(\frac{k}{n}\right)$$

where  $\mu$  is any partition with at least one untwisted part.

The bilinear relations (R-2) were derived as equation (5) in [Zon12]. The main result of that paper is the following.

**Theorem 3.6** ([Zon12]). *Relations (R-2) uniquely determine  $\tilde{V}_\nu^\bullet(0)$  from rubber integrals and the partial evaluations of Corollary 3.3.*

Theorem 3.6 is proven in [Zon12] by interpreting the rubber integrals in terms of double Hurwitz numbers. In Section 3.4, we reinterpret the rubber integrals as *wreath* Hurwitz numbers which allows us to utilize the representation theory of the generalized symmetric group.

**3.4. Wreath Hurwitz Numbers.** In the non-orbifold case, it was shown in [LLZ04, MLZ07] that certain rubber integrals can be interpreted in terms of double Hurwitz numbers. In this section, we generalize their result to the orbifold case.

Hurwitz numbers classically count degree  $d$  ramified covers of Riemann surfaces with monodromy around the branch points prescribed by conjugacy classes in  $S_d$ . (Cyclic) wreath Hurwitz numbers are defined to be analogous counts of degree  $dn$  ramified covers where the monodromy is prescribed by conjugacy classes  $\mu$  in  $\mathbb{Z}_n \wr S_d$ . Since  $\mathbb{Z}_n$  is in the center of  $\mathbb{Z}_n \wr S_d$ , such covers have a natural  $\mathbb{Z}_n$  action and the quotient is a classical Hurwitz cover, the monodromy is given by the underlying partitions  $\underline{\mu}$ .

We define now the particular wreath Hurwitz numbers which arise in our context.

**Definition 3.7.** Let  $H_{\nu, \mu}^{g, \gamma}$  be the automorphism-weighted count of wreath Hurwitz covers  $f : C \rightarrow \mathbb{P}^1$  where the branch locus consists of a set of  $|\gamma|$  marked punctures,  $r$  unmarked punctures, 0, and  $\infty$ , and the maps satisfy the following conditions:

- The quotient  $C/\mathbb{Z}_n$  is a connected genus  $g$  curve,
- The monodromy around 0 and  $\infty$  is given by  $\nu$  and  $\mu$ ,
- The monodromy around the branch point corresponding to  $\gamma_i \in \gamma$  is given by the conjugacy class  $(\gamma_i, 1, \dots, 1)$ ,
- The monodromy around the  $r$  additional branch points is given by the conjugacy class  $(2, 1, \dots, 1)$ .

**Remark 3.8.** Here we use the vector notation for  $n$ -tuples of partitions introduced in Section 2.1.

**Remark 3.9.**  $H_{\nu, \mu}^{g, \gamma} = 0$  unless  $r = 2g - 2 + l(\nu) + l(\mu)$ . In what follows, we define  $r$  to be this value.

The next theorem relates the rubber integrals which arose in the localization computations to the wreath Hurwitz numbers  $H_{\nu, \mu}^{g, \gamma}$ .

**Theorem 3.10.**

$$H_{\nu, \mu}^{g, \gamma} = \frac{r!}{|Aut(\nu)||Aut(\mu)|} \int_{\overline{\mathcal{M}}_{g, \gamma}(\mathcal{G}_0; \nu[0], \mu[\infty]) / \mathbb{C}^*} \psi_0^{r-1+|\gamma|}.$$

*Proof.* Via the forgetful map  $F : \overline{\mathcal{M}}_{g, \gamma}(\mathcal{G}_0; \nu[0], \mu[\infty]) \rightarrow \overline{\mathcal{M}}_{g, n}(\mathbb{P}^1; \underline{\nu}[0], \underline{\mu}[\infty])$ , we obtain a branch morphism  $Br : \overline{\mathcal{M}}_{g, \gamma}(\mathcal{G}_0; \nu[0], \mu[\infty]) \rightarrow \text{Sym}^r \mathbb{P}^1 \cong \mathbb{P}^r$  by postcomposing  $F$  with the usual branch morphism. For each of the  $n$  (twisted) marked points, we also obtain maps  $\tilde{ev}_i : \overline{\mathcal{M}}_{g, \gamma}(\mathcal{G}_0; \nu[0], \mu[\infty]) \rightarrow \mathbb{P}^1$  by postcomposing the usual evaluation map with the natural map to  $\mathbb{P}^1$ . Then the wreath Hurwitz numbers can be expressed as

$$(4) \quad H_{\nu, \mu}^{g, \gamma} = \frac{1}{|Aut(\nu)||Aut(\mu)|} \int_{\overline{\mathcal{M}}_{g, \gamma}(\mathcal{G}_0; \nu[0], \mu[\infty])} Br^*(pt) \cdot \prod \tilde{ev}_i^*(pt).$$

It is left to show that

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])} Br^*(pt) \cdot \prod \tilde{e}v_i^*(pt) = r! \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])/\mathbb{C}^*} \psi_0^{r-1+|\gamma|}$$

and we accomplish this via localization.

We equip the moduli space with a torus action by fixing the  $\mathbb{C}^*$  action on the target  $t \cdot [z_0 : z_1] = [z_0 : tz_1]$  so that the tangent bundle is linearized with weights 1 at  $0 = [0 : 1]$  and  $-1$  and  $\infty = [1 : 0]$ . The isomorphism  $\mathbb{P}^r = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(r))) \rightarrow \text{Sym}^r \mathbb{P}^1$  is given by  $s \rightarrow \text{Div}(s)$  where the basis  $\langle z_0^r, z_0^{r-1}z_1, \dots, z_1^r \rangle$  for  $H^0(\mathbb{P}^1, \mathcal{O}(r))$  corresponds to the homogeneous coordinates  $(y_0 : y_1 : \dots : y_r)$ . We equip  $\mathbb{P}^r$  with the torus action  $t \cdot (y_0 : y_1 : \dots : y_r) = (y_0 : ty_1 : \dots : t^r y_r)$  which makes  $Br$  an equivariant map. We lift  $[pt] \in H^{2r}(\mathbb{P}^r)$  to  $\prod_{i=0}^{r-1} (H + i\hbar) \in H_{\mathbb{C}^*}^{2r}(\mathbb{P}^r)$ , the preimage of this lift is the locus of maps where the simple ramification points map to  $\infty$ . Likewise we lift

$$\tilde{e}v_i^*(pt) = c_1(\tilde{e}v_i^*\mathcal{O}(1))$$

by linearizing  $\mathcal{O}(1)$  with weights 0 at 0 and  $-1$  at  $\infty$ .

With these choices of linearizations, we see that the integrand vanishes on all fixed loci where any of the  $n+r$  points with nontrivial monodromy map to 0. This leaves exactly one fixed locus where the target expands over  $\infty$  and everything interesting happens over the expansion. On this locus, the integrand specializes to  $(-\hbar)^{r+n}r!$  and the inverse of the equivariant Euler class of the normal bundle is

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])/\mathbb{C}^*} \frac{1}{-\hbar - \psi_0}.$$

Therefore the contribution, and hence the integral in (4), is equal to

$$r! \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])/\mathbb{C}^*} \psi_0^{r+n-1}.$$

□

**Corollary 3.11.**

$$\begin{aligned} H_{\nu,\mu}^{\bullet}(x, u) &= \exp \left( \sum_{g,\gamma} H_{\nu,\mu}^{g,\gamma} \frac{u^r}{r!} \frac{x^\gamma}{\gamma!} \right) \\ &= \sum_{g,\gamma} H_{\nu,\mu}^{X,\gamma^\bullet} \frac{u^r}{r!} \frac{x^\gamma}{\gamma!} \end{aligned}$$

where  $H_{\nu,\mu}^{X,\gamma^\bullet}$  is the wreath Hurwitz number with possibly disconnected covers.

By the Burnside formula ([Dij95]), we compute

$$H_{\nu,\mu}^{g,\gamma^\bullet} = \sum_{|\lambda|=d} (f_T(\lambda))^r \prod (f_i(\lambda))^{m_i(\gamma)} \frac{\chi_\lambda(\mu)}{z_\mu} \frac{\chi_\lambda(\nu)}{z_\nu}$$

where

$$f_T(\lambda) := \frac{nd(d-1)\chi_\lambda(2, 1, \dots, 1)}{2 \cdot \dim \lambda}$$

and

$$f_i(\lambda) := \frac{d\chi_\lambda(\xi^i, 1, \dots, 1)}{\dim \lambda}.$$

Therefore we obtain the following form for the generating function of wreath Hurwitz numbers:

$$(5) \quad H_{\nu, \mu}^\bullet(x, u) = \sum_{|\lambda|=d} \frac{\chi_\lambda(\mu)}{z_\mu} \frac{\chi_\lambda(\nu)}{z_\nu} e^{f_T(\lambda)u + \sum f_i(\lambda)x_i}.$$

Using the fact that  $\chi_\lambda(-\nu) = \overline{\chi_\lambda(\nu)}$ , orthogonality of characters gives us the following relations:

$$(6) \quad H_{\nu, \mu}^\bullet(x + y, u + v) = \sum_{\sigma} H_{\nu, \sigma}^\bullet(x, u) z_\sigma H_{-\sigma, \mu}^\bullet(y, v)$$

and

$$(7) \quad H_{\nu, -\mu}^\bullet(0, 0) = \frac{1}{z_\mu} \delta_{\nu, \mu}.$$

Using (6) and (7) to invert (R-1), we obtain the following.

**Lemma 3.12.** *Framing dependence in the conjugacy basis:*

$$\tilde{V}_\mu^\bullet(a) = \sum_{|\nu|=|\mu|} \tilde{V}_\nu^\bullet(0) z_\nu \tilde{H}_{-\nu, \mu}^\bullet(-a)$$

In particular, Lemma 3.12 shows that the relations (R-1) determine the general framed vertex from the  $a = 0$  vertex and it tells us the correspondence exactly in terms of characters of the generalized symmetric group.

Define

$$\hat{P}_\lambda(a) := \sum_{\mu} \tilde{V}_\mu^\bullet(a) \chi_\lambda(-\mu)$$

or equivalently

$$\tilde{V}_\mu^\bullet(a) = \sum_{\lambda} \hat{P}_\lambda(a) \frac{\chi_\lambda(\mu)}{z_\mu}.$$

Then Lemma 3.12 is equivalent to the following.

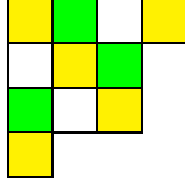
**Lemma 3.13.** *Framing dependence in the representation basis:*

$$\hat{P}_\lambda(a) = e^{-\sqrt{-1}af_T(\lambda)u-a\sum \xi_{2n}^{-i}f_i(\lambda)x_i} \hat{P}_\lambda(0)$$

#### 4. COMBINATORICS

In the previous section, we investigated the geometrically defined Gromov-Witten vertex  $\tilde{V}_\mu^\bullet(a)$ . In this section, we investigate the combinatorially defined Donaldson-Thomas vertex  $\tilde{P}_\lambda(a)$  and relate it to the loop Schur functions.

**4.1. Loop Schur Functions.** For a positive integer  $n$  and partition  $\rho$ , the *colored* Young diagram  $(\rho, n)$  is obtained by coloring the boxes of the Young diagram by their *content* modulo  $n$ . In other words if  $\square$  is in the  $i$ th row and the  $j$ th column, we color it with  $c(\square) := j - i \pmod n$ . For example, if  $\rho = (4, 3, 3, 1)$  and  $n = 3$ , the colored Young diagram is given by



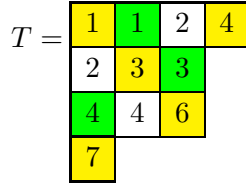
with

$$0 \leftrightarrow \text{yellow box}, \quad 1 \leftrightarrow \text{green box}, \quad \text{and} \quad 2 \leftrightarrow \text{white box}$$

We let  $\rho[i]$  denote the collection of boxes with color  $i$ . A *semi-standard Young tableau* (SSYT) of  $\rho$  is a numbering of the boxes so that numbers are weakly increasing left to right and strictly increasing top to bottom. For each SSYT  $T$  and  $\square \in \rho$ , we define the *weight*  $w(\square, T)$  to be the number appearing in that square. To each  $\rho$ ,  $n$ , and  $T \in SSYT(\rho, n)$ , we associate a monomial in  $n$  infinite sets of variables  $\{q_{i,j} | i \in \mathbb{Z}_n, j \in \mathbb{N}\}$ :

$$q^T := \prod_{i=0}^{n-1} \prod_{\square \in \rho[i]} q_{i,w(\square, T)}.$$

For example, the SSYT



gives the monomial

$$q^T = q_{0,1} q_{0,3} q_{0,4} q_{0,6} q_{0,7} q_{1,1} q_{1,3} q_{1,4} q_{2,2}^2 q_{2,4}.$$

**Definition 4.1.** The *loop Schur function* associated to  $(\rho, n)$  is defined by

$$s_\rho[n] := \sum_{T \in SSYT(\rho, n)} q^T.$$

**Remark 4.2.** In the current setting, we only care about the case where  $\rho = \bar{\lambda}$  arises from an  $n$ -tuple of partitions  $\lambda$ . This is equivalent to the condition  $|\rho[i]| = |\rho[j]|$  for all  $i, j$ .

Denote by  $S_\lambda$  the function in  $n$  variables obtained by making the substitution  $q_{i,j} = q_i^j$  in  $s_{\bar{\lambda}}[n]$ . The following result appears in both [ER88] and [Nak09].

**Lemma 4.3** ([ER88, Nak09]).

$$S_\lambda = \frac{\prod_i q_i^{n_i(\bar{\lambda})}}{\prod_{\square \in \bar{\lambda}} \left(1 - \prod_i q_i^{h_i(\square)}\right)}.$$

As a consequence, we have the following identity:

**Corollary 4.4.**

$$\tilde{P}_\lambda(0) = \frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\lambda)} q^{\frac{d}{2}} (-1)^d S_\lambda.$$

We also recall the definition of the series  $s_\rho^k[n]$  from [Ros12]. For  $0 \leq k < n$ , define the shifted weight

$$w^k(\square, T) := w(\square, T) + \frac{k \cdot c(\square)}{n}$$

and the corresponding monomial

$$q^{T,k} := \prod_{i=0}^{n-1} \prod_{\square \in \rho[i]} q_{i,w^k(\square,T)}$$

where the second index belongs to  $\frac{1}{n}\mathbb{Z}$ .

**Definition 4.5.** The  $k$ -shifted Schur function associated to  $(\rho, n)$  is

$$s_\rho^k[n] := \sum_{T \in SSYT(\rho, n)} q^{T,k}.$$

We denote by  $S_\lambda^k$  the series in  $n$  variables obtained from  $s_\rho^k[n]$  by specializing  $q_{i,j} = q_i^j$ . We have the following natural generalization of Corollary 4.4.

**Lemma 4.6.**

$$\tilde{P}_\lambda(0) = \frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\lambda)} q^{\frac{d}{2}} (-1)^d S_\lambda^k \left( \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} \right)^{\frac{-k}{n}}.$$

**4.2. Combinatorial Identities.** The following (specializations of) results from [Ros12] will be integral in proving Theorem 1.

**Theorem 4.7** ([Ros12], Theorem 1).

$$\frac{1}{1 - (q_0 \dots q_{n-1})^l} S_\lambda = \sum (-1)^{ht(\bar{\sigma} \setminus \bar{\lambda}) - 1} S_\sigma$$

where the sum is over all ways of adding a connected length  $ln$  border strip to  $\bar{\lambda}$  and  $ht(\bar{\sigma} \setminus \bar{\lambda})$  is the number of rows that the border strip occupies.

**Theorem 4.8** ([Ros12], Theorem 2). For a fixed  $\bar{\lambda}$  and  $k \neq 0$ ,

$$\sum (-1)^{ht(\bar{\sigma} \setminus \bar{\lambda}) - 1} S_\sigma^k = 0$$

where the sum is over all ways of adding a connected length  $ln$  border strip to  $\bar{\lambda}$ .

## 5. REPRESENTATION THEORY

In this section we investigate certain characters of the generalized symmetric group which arose in Section 3.4. Our main tool is the wreath Fock space. We begin by recalling the basic definitions and results concerning the usual Fock space.

**5.1. The Infinite Wedge.** The infinite wedge provides a convenient setting for studying the representation theory of the symmetric group in terms of combinatorial manipulations of partitions and Maya diagrams. For a more thorough treatment of the infinite wedge and some of its applications in Gromov-Witten theory, see for example [OP02, OP04] or for an application in double Hurwitz numbers, see [Joh10].

Let  $V$  be the infinite vector space with spanning set indexed by half integers:

$$V := \bigoplus_{i \in \mathbb{Z}} \left\langle \underline{i + \frac{1}{2}} \right\rangle_{\mathbb{C}}.$$

**Definition 5.1.** The *infinite wedge*  $\bigwedge^{\frac{\infty}{2}} V$  is the vector space

$$\bigwedge^{\frac{\infty}{2}} V := \bigoplus_{(i_k)} \langle \underline{i_1} \wedge \underline{i_2} \wedge \dots \rangle$$

where  $(i_k)$  is a decreasing sequence of half integers such that

$$i_k + k - \frac{1}{2} = c$$

for some constant  $c$  and  $k \gg 0$ . We call  $c$  the *charge* of the vector.

We will only be concerned with the subvector space spanned by vectors of charge 0. We denote this space by  $\bigwedge_0^{\frac{\infty}{2}} V$ .

**5.1.1. Maya Diagrams.** The primary combinatorial tool for us will be Maya diagrams. A Maya diagram can be thought of as a collection of stones placed at the half integers such that half integers without stones are bounded below and the half integers with stones are bounded above.

The basis vectors of  $\bigwedge_0^{\frac{\infty}{2}} V$  can be identified with Maya diagrams canonically as follows. Let  $S = \{i_k\}$  where  $(i_k)$  corresponds to a charge 0 vector. Then  $m_S$  is the Maya diagram with a stone in the  $i$ th place if and only if  $i \in S$ .

**5.1.2. Partitions.** The charge zero basis vectors can also be canonically identified with partitions. If we let  $\alpha$  be the increasing sequence of half integers in  $S \cap \mathbb{Q}_{>0}$  and  $\beta$  the increasing sequence of half integers in  $-(S^c \cap \mathbb{Q}_{<0})$ , then  $(\alpha|\beta)$  is the modified Frobenius coordinate of a partition  $\rho$ . In other words, representing  $\rho$  as a Young diagram,  $\alpha_i$  is the number of boxes (half-boxes included) in the  $i$ th row to the right of the main diagonal and  $\beta_i$  is the number of boxes in the  $i$ th column below the main diagonal.

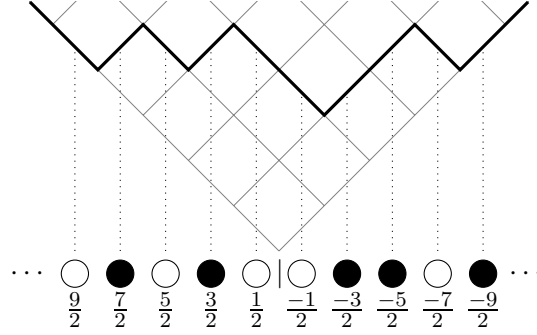


FIGURE 1. Correspondence between the different combinatorial bases of  $\bigwedge_0^{\infty} V$ .

Equivalently, the partition  $\rho = (\rho_1, \rho_2, \dots)$  is determined by writing  $v_S$  in the following form.

$$v_S = \underline{\rho_1 - 1/2} \wedge \underline{\rho_2 - 3/2} \wedge \dots$$

To relate partitions to Maya diagrams, rotate the corresponding Young diagram counterclockwise by 135 and place 0 directly below the vertex. The stones in the Maya diagram lie directly below outer edges of the Young diagram which have slope 1. This correspondence is illustrated in Figure 1.

5.1.3. *One Basis.* With the above correspondences, we will think of  $\bigwedge_0^{\infty} V$  simultaneously as the vector space spanned by

- Sequences  $S$  of the half integers with charge 0,
- Maya diagrams, or
- Partitions.

For simplicity, we will denote the basis elements by  $v_\rho$  keeping in mind that the partition  $\rho$  corresponds canonically to a Maya diagram  $m_\rho$  and a set of half integers  $S_\rho$ .

5.1.4. *Operators.* In order to relate the apply the infinite wedge to our study of representation theory, we define several operators on  $\bigwedge_0^{\infty} V$  via their action on basis elements  $v_\rho$ .

For any half integer  $k$  and basis element  $v_\rho$ , the operator  $E_{k,k}$  acts on  $v_\rho$  as follows:

$$E_{k,k}v_\rho = \begin{cases} v_\rho & k > 0, k \in S \\ -v_\rho & k < 0, k \notin S \\ 0 & \text{else.} \end{cases}$$



For  $k$  a positive integer, the creation operator  $\alpha_{-k}$  acts on  $v_\rho$  as follows:

$$\alpha_{-k}v_\rho = \sum_{\tau} (-1)^{ht(\tau \setminus \rho)} v_\sigma$$

where the sum is over all ways of adding a  $k$ -strip to  $\rho$ . In terms of Maya diagrams, the sum is over all ways of moving a stone  $k$  places to the left and the sign corresponds to the number of stones jumped during such a move.

To relate the infinite wedge to the representation theory of the symmetric group, recall that each partition  $\rho$  corresponds to an irreducible representation of  $S_d$  with character  $\chi_\rho$ . Given a partition  $\tau = (d_1, \dots, d_l)$  corresponding to a conjugacy class in  $S_d$ , we define the operator

$$\alpha_{-\tau} := \prod_{i=1}^l \alpha_{-d_i}$$

The following identity follows from the Murnaghan-Nakayama formula for characters of the symmetric group:

$$(8) \quad \alpha_{-\tau}v_\emptyset = \sum_{\rho} \chi_\rho(\tau)v_\rho.$$

We also define the operator

$$\mathcal{F}_T := \sum_k \frac{k^2}{2} E_{k,k}.$$

If  $T$  is the conjugacy class of a transposition, then the eigenvalues of  $\mathcal{F}_T$  are the central characters  $f_T(\lambda) := \frac{|T|\chi_\lambda(T)}{\dim(\lambda)}$ :

$$(9) \quad \mathcal{F}_T \cdot v_\lambda = f_T(\lambda)v_\lambda.$$

**5.2. Wreath Fock Space.** The wreath product generalization of the Fock space gives a combinatorial tool for manipulating the representation theory of the groups  $G \wr S_d$ . These spaces and their corresponding operators have been developed in e.g. [FW01, QW04, Joh09]. We merely focus on the cyclic case which is all we require. To that end, the wreath Fock space can be defined as

$$\mathcal{Z}_n := \bigotimes_{\{0, \dots, n-1\}} \bigwedge_0^{\frac{\infty}{2}} V.$$

Basis vectors correspond to  $n$ -tuples of partitions  $\lambda = (\lambda_0, \dots, \lambda_{n-1})$  or, equivalently,  $n$ -tuples of Maya diagrams.

In the wreath Fock space, there is an additional way by which we will distinguish a basis element. Given an  $n$ -tuple of Maya diagrams, we can interlace them to get a single Maya diagram by sending a stone in the  $k$ th place of the  $i$ th Maya diagram to position  $n(k - \frac{1}{2}) + (i + \frac{1}{2})$  in the new Maya diagram. An example of this identification is shown in Figure 2. This new Maya diagram corresponds to a partition of  $nd$  which we denote  $\bar{\lambda}$ . Reversing this process is usually referred to as an  $n$ -quotient. As in Section



For  $\mu = (1^d)$ , the coefficient of  $v_\lambda$  in (10) can be interpreted as the number of ways to build  $\lambda = (\lambda_0, \dots, \lambda_{n-1})$  one box at a time. Equivalently, this can be interpreted as the number of standard Young tableaux of  $\lambda$ , i.e. the number of ways to fill the boxes of the  $\lambda_i$  with the numbers  $1, \dots, d$  with the property that numbers always increase along rows and down columns. This is easily computed:

$$(12) \quad \chi_\lambda(1^d) = \binom{d}{d_0, \dots, d_{n-1}} \prod \dim(\lambda_i)$$

where we use the fact that  $\dim(\lambda_i)$  is the number of standard tableaux of  $\lambda_i$ .

On the other hand, for  $\mu = (\xi^i, 1^{d-1})$ , the coefficient of  $v_\lambda$  in (10) can be interpreted as a weighted count of ways to build  $\lambda$  one box at a time, where the weight is  $\xi^{-ij}$  if the first box is a part of  $\lambda_j$ . This is also easily computed:

$$(13) \quad \chi_\lambda(\xi^i, 1^{d-1}) = \sum_{j=0}^{n-1} \xi_n^{-ij} \binom{d-1}{d_0, \dots, d_j-1, \dots, d_{n-1}} \prod \dim(\lambda_i).$$

Identity (i) follows by dividing (13) by (12) and multiplying by  $d$ .

To prove identity (ii), begin by writing  $\bar{\lambda} = (\alpha|\beta)$  in modified Frobenius notation (c.f. Section 5.1). Then the number of boxes in  $\bar{\lambda}[0]$  to the right (below) the  $i$ th diagonal element is given by  $\lfloor \frac{\alpha_i}{n} \rfloor$   $\left( \lfloor \frac{\beta_i}{n} \rfloor \right)$ . If we compute the sum in (ii) over these  $\lfloor \frac{\alpha_i}{n} \rfloor$   $\left( \lfloor \frac{\beta_i}{n} \rfloor \right)$  terms, we get a contribution of

$$n + 2n + \dots + n \left\lfloor \frac{\alpha_i}{n} \right\rfloor \quad \left( -n - 2n - \dots - n \left\lfloor \frac{\beta_i}{n} \right\rfloor \right).$$

Therefore, the right side of the (ii) can be written as

$$(14) \quad \sum_{(i,j) \in \bar{\lambda}[0]} j - i = n \sum_{i=1}^{\infty} \left( \frac{\lfloor \frac{\alpha_i}{n} \rfloor^2 + \lfloor \frac{\alpha_i}{n} \rfloor}{2} - \frac{\lfloor \frac{\beta_i}{n} \rfloor^2 + \lfloor \frac{\beta_i}{n} \rfloor}{2} \right).$$

To compute the left side of (ii), we consider equation (11). Via the  $n$ -quotient correspondence described above, we can interpret  $v_\lambda$  as a vector  $v_{\bar{\lambda}} \in \Lambda_0^{\frac{\infty}{2}}$ . Under this correspondence, the operator  $n \sum_{i=0}^{n-1} \mathcal{F}_T^i$  becomes

$$n \sum_k \frac{1}{2} \left( \left\lfloor \frac{k}{n} \right\rfloor + \frac{1}{2} \right)^2 E_{kk}.$$

Each summand acts simply by multiplying  $v_{\bar{\lambda}}$  by an appropriate scalar. This scalar is zero unless  $k = \alpha_i > 0$  or  $k = -\beta_i < 0$  for some  $i$ . In these cases, the scalar is

$$n \frac{1}{2} \left( \left\lfloor \frac{\alpha_i}{n} \right\rfloor + \frac{1}{2} \right)^2$$

and

$$-n \frac{1}{2} \left( \left\lfloor \frac{\beta_i}{n} \right\rfloor + \frac{1}{2} \right)^2.$$

We obtain (14) by summing over all such  $i$ .  $\square$

**Lemma 5.3.** *After the change of variables prescribed by Theorem 1,*

$$(15) \quad \left( \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} \right)^{1/n} = (-\xi_{2n})^{-d} \left( \xi_n^{-\sum id_i} \right) e^{\frac{1}{n}(\sqrt{-1}f_T(\lambda)u + \sum \xi_{2n}^{-k} f_k(\lambda)x_k)}$$

*Proof.* If  $\lambda = (\lambda_0, \dots, \lambda_{n-1})$  with  $|\lambda_i| = d_i$ , then in terms of Maya diagrams we can interpret the  $d_i$  as follows:  $d_i$  is the number of moves it takes to build the Maya diagram of  $\lambda_i$  from the empty Maya diagram by only moving stones one place at a time. Moreover, each such move has the effect of adding a length  $n$  border strip to  $\bar{\lambda}$ , the northeast-most box in the strip having color  $i$ . Moreover, the quantity  $j - i$  decreases uniformly by 1 as we move south and west along the strip. Therefore, each such move contributes to  $\prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i}$  a factor of

$$q_i^k q_{i-1}^{k-1} \dots q_1^{k-i+1} q_0^{k-i} q_{n-1}^{k-i-1} \dots q_{i+1}^{k-n+1} = q^{k-i} (q_i^i q_{i-1}^{i-1} \dots q_1^1 q_{n-1}^{-1} \dots q_{i+1}^{i-n+1})$$

for some  $k$ . Combining these factors, we find

$$(16) \quad \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} = q^M (q_{n-1}^{-1} \dots q_1^{-n+1})^{d_0} \prod_{i=1}^{n-1} (q_i^i q_{i-1}^{i-1} \dots q_1^1 q_{n-1}^{-1} \dots q_{i+1}^{i-n+1})^{d_i}$$

for some value  $M$ . But  $M$  is  $\sum_{(i,j) \in \bar{\lambda}[0]} (j - i)$  which we know is equal to  $f_T(\lambda)$  from Lemma 5.2.

It is left to investigate what happens to the factors in (16) after the change of variables. Since  $q \rightarrow e^{\sqrt{-1}u}$  and  $M = f_T(\lambda)$ , then we see immediately that the  $u$  factors on either side of (15) agree.

We now compute the coefficient of  $d_i x_j$  in the exponent of (16) after the change of variables. To do this, we must compute the coefficient of  $x_j$  in the factor  $q_i^i q_{i-1}^{i-1} \dots q_1^1 q_{n-1}^{-1} \dots q_{i+1}^{i-n+1}$ . Applying the change of variables, this coefficient is

$$(17) \quad - \sum_{r=1}^i \frac{r \xi_n^{-jr}}{n} (\xi_{2n}^j - \xi_{2n}^{-j}) - \sum_{s=i+1}^{n-1} \frac{(s-n) \xi_n^{-js}}{n} (\xi_{2n}^j - \xi_{2n}^{-j}).$$

Many of the terms in the sum cancel. Using the fact that the roots of unity sum to zero, (17) simplifies to  $\xi_{2n}^{j(-2i-1)}$ . A similar computation shows that the coefficient of  $d_0 x_j$  is  $\xi_{2n}^{-j}$ . Therefore, the coefficient of  $x_j$  is

$$\xi_{2n}^{-j} \sum \xi_n^{-ij} d_i = \xi_{2n}^{-j} f_j(\lambda)$$

where the equality follows from the first identity of Lemma 5.2.

Finally, we consider the roots of unity which appear outside of the exponent in the change of variables. The root of unity that factors out of

the term  $(q_{n-1}^{-1} \dots q_1^{-n+1})^{1/n}$  is easily computed to be  $-\xi_{2n}^{-1}$  and the root of unity which factors out of the term  $(q_i^i q_{i-1}^{i-1} \dots q_1^1 q_{n-1}^{-1} \dots q_{i+1}^{i-n+1})^{1/n}$  is  $-\xi_{2n}^{-1} \xi_n^{-i}$ . Putting all of this together proves the result.  $\square$

**5.4. Signs.** We will need a few more representation theoretic lemmas, we first introduce some notation.

If  $\bar{\sigma}$  is obtained from  $\bar{\lambda}$  by adding a length  $kn$  border strip, then the  $n$ -tuple of Maya diagrams corresponding to  $\sigma$  is obtained from those corresponding to  $\lambda$  by moving a stone  $k$  places in the  $i$ th row. Let  $\beta(\sigma \setminus \lambda)$  denote the number of stones in the  $i$ th row which are skipped over. Equivalently,  $(-1)^{\beta(\sigma \setminus \lambda)}$  is the coefficient of  $v_\sigma$  in  $\alpha_{-k}^i(v_\lambda)$ .

The first lemma allows us to deal with the sign  $\frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\bar{\lambda})}$  appearing in Theorem 1.

**Lemma 5.4.** *If  $\bar{\sigma}$  is obtained from  $\bar{\lambda}$  by adding a length  $kn$  border strip beginning with a color  $i$  box, then*

$$\frac{\chi_{\bar{\sigma}}(n^{d+k})}{\dim(\sigma)} = (-1)^{\beta(\sigma \setminus \lambda) + ht(\bar{\sigma} \setminus \bar{\lambda}) - 1} \frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\bar{\lambda})}.$$

*Proof.* By (8),  $\chi_{\bar{\lambda}}(n^d)$  is the weighted sum of ways to create the Maya diagram of  $\bar{\lambda}$  from the vacuum diagram by moving stones  $n$  places at a time; the weight is  $\pm 1$  depending on whether the total number of stones jumped over is even or odd. It is not hard to see that the weight of any such sequence is equal to the weight of any other. Since  $\dim(\bar{\lambda})$  is the total number of such sequences, we see that  $\frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\bar{\lambda})}$  is equal to the weight of any one of them.

Now suppose  $\bar{\sigma}$  is obtained from  $\bar{\lambda}$  by adding a length  $kn$  border strip. We can think of  $\bar{\sigma}$  as being obtained from  $\bar{\lambda}$  by moving a single stone  $kn$  places to the left in the Maya diagram of  $\bar{\lambda}$ ,  $ht(\bar{\sigma} \setminus \bar{\lambda}) - 1$  is the total number of stones jumped while  $\beta(\sigma \setminus \lambda)$  counts the number of jumped stones which are  $n, 2n, 3n, \dots$  positions to the left of where the stone sat originally.

On the other hand, the Maya diagram of  $\bar{\sigma}$  can be obtained from that of  $\bar{\lambda}$  by choosing a sequence of length  $n$  jumps. As above,  $\frac{\chi_{\bar{\sigma}}(n^{d+k})}{\dim(\sigma)} = (-1)^* \frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\bar{\lambda})}$  where  $*$  is equal to the total number of stones jumped during the sequence of moves. With the above interpretations for  $ht(\bar{\sigma} \setminus \bar{\lambda}) - 1$  and  $\beta(\sigma \setminus \lambda)$ , we see that the number of stones jumped in this process is  $(ht(\bar{\sigma} \setminus \bar{\lambda}) - 1) - \beta(\sigma \setminus \lambda)$ .  $\square$

The final lemma of this section allows us to compare  $\chi_\lambda(\mu)$  with  $\chi_\lambda(g_k(\mu))$ .

**Lemma 5.5.** *If  $\lambda = (\lambda_0, \dots, \lambda_{n-1})$  with  $|\lambda_j| = d_j$ , then*

$$\chi_\lambda(g_k(\mu)) = \xi_n^{-k \sum j d_j} \chi_\lambda(-\mu).$$

*Proof.* Write  $\mu = (\mu^0, \dots, \mu^{n-1})$  with  $\mu^s = (d_1^s, \dots, d_{l_s}^s)$  and define  $(,)$  to be the inner product for which  $\{v_\lambda\}$  is an orthonormal basis. By (10), we have

$$\begin{aligned}\chi_\lambda(g_k(\mu)) &= \left( \prod_{s=0}^{n-1} \prod_{i=0}^{l_s} \left( \sum_{j=0}^{n-1} \xi_n^{-d_i^s k j + s j} \alpha_{-d_i^s}^j \right) v_\emptyset, v_\lambda \right) \\ &= \xi_n^{-k \sum j d_j} \left( \prod_{s=0}^{n-1} \prod_{i=0}^{l_s} \left( \sum_{j=0}^{n-1} \xi_n^{s j} \alpha_{-d_i^s}^j \right) v_\emptyset, v_\lambda \right) \\ &= \xi_n^{-k \sum j d_j} \chi_\lambda(-\mu).\end{aligned}$$

□

## 6. PROOF OF THEOREM 1

Theorem 3.6, equation (5), and Lemma 3.13 justify the following reduction.

**Reduction 6.1.** *To prove Theorem 1, it suffices to check that the following properties hold after the prescribed change of variables.*

(I) *The framing factors are consistent:*

$$\left( \left( (-\xi_{2n})^{|\lambda|} \prod \xi_n^{l|\lambda_l|} \right)^n \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} \right)^a = e^{\sqrt{-1} a f_T(\lambda) u + a \sum \xi_{2n}^{-i} f_i(\lambda) x_i}.$$

(II)  $\tilde{P}_\lambda(0)$  *satisfy the partial evaluations of Corollary 3.3:*

$$\sum_{|\lambda|=|\mu|} \tilde{P}_\lambda(0) \frac{\chi_\lambda(\mu)}{z_\mu} = \left( \frac{1}{z_{\mu^0}} \prod_{j=1}^{l_0} \frac{\sqrt{-1}(-1)^{d_j^0}}{2} \csc \left( \frac{d_j^0 u}{2} \right) \right) \left( \sum_{|\sigma|=|\mu'|} \tilde{P}_\sigma(0) \frac{\chi_\sigma(\mu')}{z_{\mu'}} \right).$$

(III)  $\tilde{P}_\lambda(0)$  *satisfy the relations (R-2) for all  $\mu$  with at least one untwisted part:*

$$\sum_\nu \left( \sum_\lambda \tilde{P}_\lambda(0) \frac{\chi_\lambda(\nu)}{z_\nu} \right) z_\nu \left( \sum_\sigma \frac{\chi_\sigma(g_k(\nu))}{z_{g_k(\nu)}} \frac{\chi_\sigma(\mu)}{z_\mu} e^{\frac{k}{n}(\sqrt{-1} f_T(\sigma) u + \sum \xi_{2n}^{-i} f_i(\sigma) x_i)} \right) = 0.$$

We now check identities (I) - (III).

**Identity (I).** This follows immediately from Lemma 5.3.

**Identity (II).** Since  $z_\mu = z_{\mu^0} z_{\mu'}$ , we must show that

$$\sum_{|\lambda|=|\mu|} \tilde{P}_\lambda(0) \chi_\lambda(\mu) = \left( \prod_{j=1}^{l_0} \frac{\sqrt{-1}(-1)^{d_j^0}}{2} \csc \left( \frac{d_j^0 u}{2} \right) \right) \left( \sum_{|\sigma|=|\mu'|} \tilde{P}_\sigma(0) \chi_\sigma(\mu') \right).$$

after the change of variables. To do this, it is equivalent to show

$$\sum_{|\lambda|=|\mu|+k} \tilde{P}_\lambda(0) \chi_\lambda(\mu \cup (k)) = \frac{\sqrt{-1}(-1)^k}{2} \csc\left(\frac{ku}{2}\right) \left( \sum_{|\sigma|=|\mu|} \tilde{P}_\sigma(0) \chi_\sigma(\mu) \right)$$

which is equivalent (before the change of variables) to

$$(18) \quad \sum_{|\lambda|=|\mu|+k} \tilde{P}_\lambda(0) \chi_\lambda(\mu \cup (k)) = \frac{(-1)^k q^{\frac{k}{2}}}{1-q^k} \sum_{|\sigma|=|\mu|} \tilde{P}_\sigma(0) \chi_\sigma(\mu).$$

Fix  $\sigma$ . Then

$$\begin{aligned} \frac{(-1)^k q^{\frac{k}{2}}}{1-q^k} \tilde{P}_\sigma(0) \chi_\sigma(\mu) &= \frac{(-1)^{k+|\mu|} q^{\frac{k}{2}}}{1-q^k} \chi_\sigma(\mu) \frac{\chi_{\bar{\sigma}}(n^{|\mu|})}{\dim(\sigma)} q^{\frac{|\mu|}{2}} S_\sigma \\ &= (-1)^{k+|\mu|} q^{\frac{|\mu|+k}{2}} \chi_\sigma(\mu) \frac{\chi_{\bar{\sigma}}(n^{|\mu|})}{\dim(\sigma)} \sum_{\bar{\lambda} \supset \bar{\sigma}} (-1)^{ht(\bar{\lambda} \setminus \bar{\sigma})-1} S_{\bar{\lambda}} \\ &= \chi_\sigma(\mu) \sum_{\bar{\lambda} \supset \bar{\sigma}} (-1)^{\beta(\lambda \setminus \sigma)} \frac{\chi_{\bar{\lambda}}(n^{|\mu|+k})}{\dim(\lambda)} q^{\frac{|\lambda|}{2}} (-1)^{|\lambda|} S_\lambda \\ (19) \quad &= \chi_\sigma(\mu) \sum_{\bar{\lambda} \supset \bar{\sigma}} (-1)^{\beta(\lambda \setminus \sigma)} \tilde{P}_\lambda(0). \end{aligned}$$

where the sum is over all  $\bar{\lambda}$  obtained from  $\bar{\sigma}$  by adding a length  $kn$  border strip. The first equality follows from Corollary 4.4, the second from Theorem 4.7, the third from Lemma 5.4, and the fourth is another application of Corollary 4.4.

From (10), we know

$$(20) \quad \chi_\lambda(\mu \cup (k)) = \sum_{\sigma} \chi_\sigma(\mu) (-1)^{\beta(\lambda \setminus \sigma)},$$

where the sum is over all  $\sigma$  such that  $\bar{\sigma}$  is obtained from  $\bar{\lambda}$  by removing a  $kn$  strip. Summing (19) over all  $\sigma$  proves identity (18) and thus (II).

**Identity (III).** Applying Lemma 5.5, (III) is equivalent to

$$\sum_{\nu} \left( \sum_{\lambda} \tilde{P}_\lambda(0) \frac{\chi_\lambda(\nu)}{z_\nu} \right) z_\nu \left( \sum_{\sigma} \xi_n^{-k \sum j|\sigma_j|} \frac{\chi_\sigma(-\nu)}{z_\nu} \frac{\chi_\sigma(\mu)}{z_\mu} e^{\frac{k}{n}(\sqrt{-1}f_T(\sigma)u + \sum \xi_{2n}^{-i} f_i(\sigma)x_i)} \right) = 0.$$

Summing over all  $\nu$  and using orthogonality of characters, the left side becomes

$$\sum_{\lambda} \tilde{P}_\lambda(0) \frac{\chi_\lambda(\mu)}{z_\mu} \xi_n^{-k \sum j|\lambda_j|} e^{\frac{k}{n}(\sqrt{-1}f_T(\lambda)u + \sum \xi_{2n}^{-i} f_i(\lambda)x_i)}.$$

Applying Lemma 5.3, we then see that (III) is equivalent to

$$\sum_{\lambda} \tilde{P}_\lambda(0) \chi_\lambda(\mu) \left( \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} \right)^{k/n} = 0$$

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for any  $\mu$  with at least one untwisted part. This is equivalent to

$$(21) \quad \sum_{\lambda} \tilde{P}_{\lambda}(0) \chi_{\lambda}(\nu \cup (k)) \left( \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} \right)^{k/n} = 0$$

for any  $\nu$ . Fix  $\sigma$  with  $|\sigma| = |\nu|$ . Then

$$\begin{aligned} 0 &= \sum_{\bar{\lambda} \supset \bar{\sigma}} (-1)^{ht(\bar{\lambda} \setminus \bar{\sigma})-1} S_{\lambda}^k \\ &= \sum_{\bar{\lambda} \supset \bar{\sigma}} \frac{\chi_{\bar{\sigma}}(n^{|\sigma|})}{\dim(\sigma)} \chi_{\sigma}(\nu) (-1)^{ht(\bar{\lambda} \setminus \bar{\sigma})-1} S_{\lambda}^k \\ &= \sum_{\bar{\lambda} \supset \bar{\sigma}} (-1)^{\beta(\lambda \setminus \sigma)} \chi_{\sigma}(\nu) q^{-\frac{|\lambda|}{2}} (-1)^{|\lambda|} \tilde{P}_{\lambda}(0) \left( \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} \right)^{k/n}. \end{aligned}$$

where the first equality is Theorem 4.8, the second holds because  $\sigma$  is fixed, and the third follows from Lemmas 4.6 and 5.4. Since  $|\lambda|$  is constant over the sum, it follows that

$$0 = \chi_{\sigma}(\nu) \sum_{\bar{\lambda} \supset \bar{\sigma}} (-1)^{\beta(\lambda \setminus \sigma)} \tilde{P}_{\lambda}(0) \left( \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{j-i} \right)^{k/n}.$$

Summing over all  $\sigma$  (using equation (20)) proves (21) and thus finishes the proof of Theorem 1.

## 7. GW/DT FOR LOCAL $\mathbb{Z}_n$ -GERBES OVER $\mathbb{P}^1$

We conclude by giving an application of the gerby Gopakumar-Mariño-Vafa formula, we prove that the Gromov-Witten potential of any local  $\mathbb{Z}_n$ -gerbe over  $\mathbb{P}^1$  is equal to the reduced, multi-regular Donaldson-Thomas potential after an explicit change of variables.

By a *local  $\mathbb{Z}_n$ -gerbe over  $\mathbb{P}^1$* , we mean the total space of a rank two Calabi-Yau bundle  $L_1 \oplus L_2$  over some  $\mathcal{G}_k$ . The CY condition implies that  $\deg(L_1) + \deg(L_2) = -2$ . Also, since the isotropy is generically trivial, we can assume that the action of  $\mathbb{Z}_n$  on the fibers of  $L_1$  ( $L_2$ ) is by  $\xi$  ( $\xi^{-1}$ ).

Fix  $k \in \{0, \dots, n-1\}$  and set  $e := \gcd(k, n)$ . Then  $\text{Pic}(\mathcal{G}_k) = \frac{e}{n}\mathbb{Z}$ . For each  $b \in \frac{e}{n}\mathbb{Z}$  we let  $\mathcal{L}_b$  denote the corresponding orbifold line bundle. The subset of  $\text{Pic}(\mathcal{G}_k)$  where  $\mathbb{Z}_n$  acts on fibers as multiplication by  $\xi$  is given by  $\mathbb{Z} - \frac{k}{n}$ . Every local  $\mathbb{Z}_n$ -gerbe over  $\mathbb{P}^1$  is isomorphic to  $\mathcal{X}_{k,b} := \text{Tot}(\mathcal{L}_b \oplus \mathcal{L}_{-b-2})$  for some  $k \in \{0, \dots, n-1\}$  and  $b \in \mathbb{Z} - \frac{k}{n}$ .



By the gluing formula of [Ros11], the degree  $d$  Gromov-Witten potential of  $\mathcal{X}_{k,b}$  is given by

$$(22) \quad GW_d(\mathcal{X}_{k,b}) = \sum_{\mu} V_{\mu}^{\bullet}(b) z_{\mu} V_{g_k(\mu)}^{\bullet}(0) \prod_{i,j} (-1)^{d_j^i b + 1 + \delta_{0,i} + \delta_{0,(-d_j^i k - i) \bmod n} + \frac{i}{n} + \frac{(d_j^i k - i) \bmod n}{n}}.$$

where the sign is the gluing term in [Ros11].

Analyzing the modification in (2), we see that (22) is equivalent to

$$(23) \quad GW_d(\mathcal{X}_{k,b}) = (-1)^{db} \sum_{\mu} \tilde{V}_{\mu}^{\bullet}(b) z_{\mu} \tilde{V}_{g_k(\mu)}^{\bullet}(0).$$

Applying the change of variables in Theorem 1, then using Lemma 5.5 and orthogonality of characters, we find that

$$\begin{aligned} GW_d(\mathcal{X}_{k,b}) &= (-1)^{db} \sum_{\mu} \left( \sum_{\lambda} \tilde{P}_{\lambda}(b) \frac{\chi_{\lambda}(\mu)}{z_{\mu}} \right) z_{\mu} \left( \sum_{\sigma} \tilde{P}_{\sigma}(0) \frac{\chi_{\sigma}(g_k(\mu))}{z_{g_k(\mu)}} \right) \\ &= (-1)^{db} \sum_{\lambda} \xi_n^{-k \sum i |\lambda_i|} \tilde{P}_{\lambda}(b) \tilde{P}_{\lambda}(0) \end{aligned}$$

From equation (3), we see that this last expression is

$$(24) \quad \sum_{\lambda} P_{\lambda}(q_0, q_1, \dots, q_{n-1}) E_{\lambda} P_{\lambda'}(q_0, q_{n-1}, \dots, q_1)$$

where

$$E_{\lambda} := \prod_{(i,j) \in \bar{\lambda}} q_{j-i}^{(b+2)i - bj - 1} (-1)^{dnb}.$$

By the main result of [BCY10], (24) is equal to the reduced, multi-regular, degree  $d$  Donaldson-Thomas potential  $DT'_{mr,d}(\mathcal{X}_{k,b})$  after the substitution  $q_0 \rightarrow -q_0$ . This proves Theorem 2.

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